Dimensional-reduction anomaly in spherically symmetric spacetimes

P. Sutton*
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1
(Received 3 April 2000; published 24 July 2000)

In D-dimensional spacetimes which can be foliated by n-dimensional homogeneous subspaces, a quantum field can be decomposed in terms of modes on the subspaces, reducing the system to a collection of $(D-n)$-dimensional fields. This allows one to write bare D-dimensional field quantities like the Green function and the effective action as sums of their $(D-n)$-dimensional counterparts in the dimensionally reduced theory. It has been shown, however, that renormalization breaks this relationship between the original and dimensionally reduced theories, an effect called the dimensional-reduction anomaly. We examine the dimensional-reduction anomaly for the important case of spherically symmetric spaces.

PACS number(s): 04.62.+v, 11.10.Gh, 11.10.Kk

I. INTRODUCTION

Spacetimes with continuous symmetries play an important role in quantum field theory for the reason that symmetries often allow one to reduce greatly the computational difficulty of a given problem, making practical calculations feasible. Of particular interest is the separation of variables approach to solving partial differential equations in geometries with a high degree of symmetry. For example, expanding a field propagating in a spherically symmetric geometry in terms of spherical harmonics and substituting into the field equation reduces the system from that of a single field $\hat{\Phi}(t,r,\theta,\phi)$ in four dimensions to a collection of effective two-dimensional fields $\hat{\varphi}_l(t,r)$, one for each spherical harmonic $Y_{lm}(\theta,\phi)$. In principle, after solving the simpler two-dimensional problems, one can obtain quantities such as the stress tensor or the effective action for the original four-dimensional field theory by summing the corresponding results for the two-dimensional field theories over all modes.

In a previous paper [1], it was noted that separation of variables can break down when applied to quantum field theory, so that summing over the dimensionally reduced results no longer yields the corresponding quantity in four dimensions. This occurs because in quantum field theory, quantities of physical interest, such as the effective action, stress energy tensor, and square of the field operator, are divergent and must be renormalized. While the bare field can be dimensionally reduced into the sum of lower-dimensional fields, the divergent parts which are to be subtracted in four dimensions generally do not equal the sum of the corresponding divergent terms from the two-dimensional theories. As a result, one obtains an incorrect answer if one calculates a renormalized quantity in four dimensions by summing over modes of the corresponding renormalized quantities in two dimensions. This failure of dimensional reduction when applied to quantum field theories is called the dimensional-reduction anomaly.

In this paper we calculate the dimensional-reduction anomaly which occurs when a scalar field propagating in a spherically symmetric four-dimensional spacetime is decomposed into spherical harmonics and treated as a collection of two-dimensional fields. This case may be of particular importance to recent attempts to calculate the stress tensor and Hawking radiation in black-hole spacetimes using two-dimensional dilaton gravity models [2–6]. Since four-dimensional renormalized quantities will not in general equal the sum of the corresponding two-dimensional quantities, it may well be necessary to take into account the contribution of the dimensional-reduction anomaly in order to reproduce the correct results for four dimensions (see also [7]).

We begin in Sec. II with a brief discussion of dimensional reduction in spherically symmetric spacetimes. In Sec. III we examine the simple case of flat space, both to illustrate the basic idea behind the anomaly and to lay the necessary computational groundwork. In Secs. IV and V we extend our calculations to a general spherically symmetric four-dimensional space, and calculate the dimensional-reduction anomalies in $\langle\hat{\Phi}^2\rangle$ and the effective action. We conclude with a brief discussion of the possible implications of the anomaly. We work in Euclidean signature, using dimensionless units where $G=c=\hbar=1$ and the sign conventions of [8] for the definition of the curvature.

II. SPHERICAL DECOMPOSITIONS

In this section we briefly consider the dimensional reduction of a quantum field in a four-dimensional spherically symmetric space, and show how it may be reduced to a collection of two-dimensional fields.

The line element for such a space may be written as

$$\begin{aligned} ds^2 &= g_{\mu\nu}(X^\tau)dX^\mu dX^\nu \\ &= h_{ab}(x^c)dx^a dx^b+\rho^2 e^{-2\phi(x^c)}\omega_{ij}(y^k)dy^i dy^j, \end{aligned} \tag{2.1}$$

where $X^\alpha=(x^a,y^i)$, h_{ab} is an arbitrary two-dimensional metric, ω_{ij} is the metric of a two-sphere, ρ is a constant with the dimensions of length, and ϕ is known as the dilaton. The radius of a two-sphere of fixed x^a is given by $r=\rho e^{-\phi(x^a)}$.

*Email address: psutton@phys.ualberta.ca

Consider a massive scalar field propagating on the space (2.1) and obeying the field equation

$$F\hat{\Phi}(X)\equiv(\Box-m^2-V)\hat{\Phi}(X)=0, \tag{2.2}$$

where the potential V is also spherically symmetric. The corresponding Green function is a solution of the equation

$$FG(X,X')=-\delta(X,X'). \tag{2.3}$$

Knowledge of the Green function for a given quantum state allows one to calculate other expectation values of interest, such as the square of the field operator, $\langle\hat{\Phi}^2\rangle$, and the stress tensor, $\langle\hat{T}_{\mu\nu}\rangle$.

Now consider what happens if we decompose $\hat{\Phi}$ in terms of spherical harmonics $Y_{lm}(y^i)$ as follows:

$$\hat{\Phi}(X)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\hat{\varphi}_l(x^a)\frac{Y_{lm}(y^i)}{r}. \tag{2.4}$$

Substitution into Eq. (2.2) shows that $\hat{\varphi}_l$ behaves as a field propagating in the two-dimensional space with line element

$$ds^2=h_{ab}(x^c)dx^a dx^b, \tag{2.5}$$

and satisfying the field equation

$$\mathcal{F}_l\hat{\varphi}_l(x)\equiv(\Delta-m^2-V_l)\hat{\varphi}_l(x)=0. \tag{2.6}$$

Here Δ is the d'Alembertian operator for the two-dimensional metric h_{ab}, and the induced potential V_l is given by

$$V_l=V+\frac{l(l+1)}{r^2}-\Delta\phi+(\nabla\phi)^2. \tag{2.7}$$

The corresponding two-dimensional Green functions $\mathcal{G}_l$ satisfy

$$\mathcal{F}_l\mathcal{G}_l(x,x')=-\delta(x,x'), \tag{2.8}$$

and are related to G via

$$G(X,X')=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi rr'}P_l(\cos\lambda)\mathcal{G}_l(x,x'), \tag{2.9}$$

where P_l is a Legendre polynomial and λ is the angular separation of X,X'.

Since the square of the field operator is given by the coincidence limit of the Green function, Eq. (2.9) implies that the four-dimensional $\langle\hat{\Phi}^2\rangle$ can be obtained by solving the two-dimensional theory for $\langle\hat{\varphi}_l^2\rangle$:

$$\langle\hat{\Phi}^2\rangle=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi r^2}\langle\hat{\varphi}_l^2\rangle. \tag{2.10}$$

The Green function, however, diverges in the coincidence limit, and must be renormalized to yield a finite $\langle\hat{\Phi}^2\rangle$. Denoting the renormalized and divergent parts by the subscripts ''ren'' and ''div'', respectively, we have

$$\langle\hat{\Phi}^2\rangle_{\rm ren}=\lim_{X'\to X}G_{\rm ren}(X,X')=\lim_{X'\to X}[G(X,X')-G_{\rm div}(X,X')], \tag{2.11}$$

$$\langle\hat{\varphi}_l^2\rangle_{\rm ren}=\lim_{x'\to x}\mathcal{G}_{l|\rm ren}(x,x')=\lim_{x'\to x}[\mathcal{G}_l(x,x')-\mathcal{G}_{l|\rm div}(x,x')]. \tag{2.12}$$

While the bare quantities G and $\mathcal{G}_l$ are related by the mode-decomposition relation (2.9), we find that the divergent parts $G_{\rm div}$ and $\mathcal{G}_{l|\rm div}$ are not. As a result, the renormalized theories in two and four dimensions are related not by Eqs. (2.9), (2.10) but rather by

$$G_{\rm ren}(X,X')=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi rr'}P_l(\cos\lambda)\times[\mathcal{G}_{l|\rm ren}(x,x')+\Delta\mathcal{G}_l(x,x')], \tag{2.13}$$

$$\langle\hat{\Phi}^2\rangle_{\rm ren}(X)=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi r^2}[\langle\hat{\varphi}_l^2\rangle_{\rm ren}(x)+\Delta\langle\hat{\varphi}_l^2\rangle(x)], \tag{2.14}$$

where the anomalous terms are easily shown to be

$$\Delta\langle\hat{\varphi}_l^2\rangle(x)=\lim_{x'\to x}\Delta\mathcal{G}_l(x,x')\equiv\lim_{x'\to x}\left[\mathcal{G}_{l|\rm div}(x,x')-2\pi rr'\int_{-1}^{1}d(\cos\lambda)\times P_l(\cos\lambda)G_{\rm div}(X,X')\right]. \tag{2.15}$$

One can show that similar formulas hold for other renormalized quantities, such as the effective action W and the stress tensor:

$$W_{\rm ren}=\sum_{l=0}^{\infty}(2l+1)[\mathcal{W}_{l|\rm ren}+\Delta\mathcal{W}_l]; \tag{2.16}$$

$$\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi r^2}[\langle\hat{\mathcal{T}}_{\mu\nu}\rangle_{l|\rm ren}+\Delta\langle\hat{\mathcal{T}}_{\mu\nu}\rangle_l]. \tag{2.17}$$

In each case the anomaly is the difference between the divergent subtraction terms for the dimensionally reduced theory and the mode-decomposed subtraction terms for the original four-dimensional theory.

Equations (2.13)–(2.15) demonstrate that the renormalized value of a field quantity is generally not equal to the sum of the same renormalized quantities for the dimensionally reduced theory. Rather, a quantity like $\langle\hat{\Phi}^2\rangle$ can be obtained from dimensional reduction only if the contribution $\langle\hat{\varphi}_l^2\rangle$ for each mode l is modified by an extra anomalous term. This failure of dimensional reduction under renormalization is the dimensional-reduction anomaly. The remainder of this paper is devoted to explicit calculations of the anomalies in $\langle\hat{\Phi}^2\rangle$ and W for the important case of spherically symmetric geometries, described by Eq. (2.1). For further general discussion of the dimensional-reduction anomaly and the related multiplicative anomaly, the reader is referred to [1] and [9–11].

At this point some conventions on notation are in order. We need to be able to distinguish quantities such as Green functions defined in different dimensions. ''Ordinary'' letters such as G, W are used for the original four-dimensional theory, while calligraphic letters such as $\mathcal{G}$, $\mathcal{W}$ refer to dimensionally reduced quantities. The anomalous difference between A, $\mathcal{A}$ is denoted $\Delta\mathcal{A}$. All curvatures will be with respect to h unless explicitly labeled otherwise; for example, $R=R[h]$ and ${}^4R=R[g]$. As for differential operators, we understand $\Box$ to represent the d'Alembertian with respect to g, while Δ is the d'Alembertian calculated using the metric h. Single covariant derivatives will be denoted by ∇; there will be no need to distinguish the metric used. For the dilaton ϕ we understand ϕ_a, ϕ_{ab}, etc. to denote multiple two-dimensional covariant derivatives of ϕ calculated using the metric h.

III. THE DIMENSIONAL-REDUCTION ANOMALY IN FLAT SPACE

The simplest example of the dimensional reduction anomaly occurs in the spherical decomposition of a scalar field in flat space, and was originally considered in [1]. We reproduce here the main formulas, as we will require them for the generalization to curved space, and because some of the notation we use is different from that of [1].

Let us assume that the potential V vanishes inside the region of interest, and is spherically symmetric outside. In this case, the Green function for a given state is renormalized by subtracting the Green function for the Euclidean vacuum. In four dimensions the latter is

$$G_{\rm div}(X,X')=\frac{m}{4\pi^2\sqrt{2\sigma}}K_1(m\sqrt{2\sigma}), \tag{3.1}$$

where σ is one-half the square of the geodesic distance between X and X', and K_1 is a modified Bessel function. In spherical coordinates $X^\mu=(t,r,\theta,\eta)$[1] the line element is

$$ds^2=dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\eta^2), \tag{3.2}$$

and

$$2\sigma=(t-t')^2+(r-r')^2+2rr'(1-\cos\lambda), \tag{3.3}$$

where λ is the angle between X and X', given by

$$\cos\lambda=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\eta-\eta'). \tag{3.4}$$

If before renormalizing we first decompose the field $\hat{\Phi}$ into spherical harmonics as in Eq. (2.4), we will be left with an effective field $\hat{\varphi}_l$ propagating on the two-dimensional space with line element $ds^2=dt^2+dr^2$, where $t\in(-\infty,\infty)$ and $r\in[0,\infty)$. In this case σ is given by $\frac{1}{2}[(t-t')^2+(r-r')^2]$, and the theory is renormalized by subtracting the two-dimensional vacuum Green function

$$\mathcal{G}_{l|{\rm div}}(x,x')=\frac{1}{2\pi}K_0(m\sqrt{(t-t')^2+(r-r')^2}). \tag{3.5}$$

To compare the renormalization of the two- and four-dimensional theories and establish the existence of the dimensional-reduction anomaly, we decompose $G_{\rm div}$ into spherical harmonics. Defining the mode decomposition by

$$G_{\rm div}(X,X')=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi rr'}P_l(\cos\lambda)G_{{\rm div}|l}(x,x') \tag{3.6}$$

in accordance with Eq. (2.9), we have

$$G_{{\rm div}|l}(x,x')=2\pi rr'\int_{-1}^{1}d(\cos\lambda)P_l(\cos\lambda)G_{\rm div}(X,X'). \tag{3.7}$$

Inserting Eq. (3.1) into Eq. (3.7) and using the well-known integral representation for K_ν,

$$\int_0^\infty dx x^{-1-\nu}\exp\left\{-x-\frac{\alpha^2}{4x}\right\}=2\left(\frac{2}{\alpha}\right)^\nu K_\nu(\alpha), \tag{3.8}$$

the integral

$$\int_{-1}^{1}dzP_l(z)e^{p(1-z)}=(-1)^l e^p\sqrt{\frac{2\pi}{p}}I_{l+1/2}(p), \tag{3.9}$$

where $I_{l+1/2}$ is a modified Bessel function [see, e.g., [12], Vol. 2, Eq. (2.17.5.2)], and the representation

$$I_{l+1/2}(p)=\frac{1}{\sqrt{2\pi p}}\sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}\frac{1}{(2p)^k}\times[(-1)^k e^p-(-1)^l e^{-p}], \tag{3.10}$$

[see, for example, Eq. (8.467) of [13]], we obtain

[1]We denote the azimuthal coordinate by η rather than ϕ for obvious reasons.

$$G_{\mathrm{div}|l}(x,x')=\frac{1}{2\pi}\sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}\Bigg[(-1)^k\frac{[(t-t')^2+(r-r')^2]^{k/2}}{(2mrr')^k}K_k[m\sqrt{(t-t')^2+(r-r')^2}]$$
$$-(-1)^l\frac{[(t-t')^2+(r+r')^2]^{k/2}}{(2mrr')^k}K_k[m\sqrt{(t-t')^2+(r+r')^2}]\Bigg]. \tag{3.11}$$

Let us compare this result for the mode-decomposed subtraction terms from four dimensions with the subtraction term for the two-dimensional theory, Eq. (3.5). While $\mathcal{G}_{l|\mathrm{div}}$ is the free-field Green function in two dimensions, it is not difficult to verify that $G_{\mathrm{div}|l}$ is the Green function for a field propagating in the centrifugal barrier potential[2]

$$V_l=\frac{l(l+1)}{r^2}, \tag{3.12}$$

which obeys Dirichlet boundary conditions at $r=0$.

Naive renormalization in two dimensions requires subtracting $\mathcal{G}_{l|\mathrm{div}}$. We see, however, that $G_{\mathrm{div}|l}$ is the quantity that should be subtracted to yield the correct results for the renormalized four-dimensional theory.[3] If one was to ignore the anomaly and calculate $\langle\hat{\Phi}^2\rangle_{\mathrm{ren}}$ using the two-dimensional $\langle\hat{\varphi}_l^2\rangle_{\mathrm{ren}}$ as in Eq. (2.10), one would obtain incorrect results, such as nonvanishing expectation values for the vacuum state. Instead, using Eqs. (2.14), (2.15), (3.5), and (3.11) one finds that the renormalized theories in two and four dimensions are related by

$$\langle\hat{\Phi}^2\rangle_{\mathrm{ren}}(X)=\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi r^2}[\langle\hat{\varphi}_l^2\rangle_{\mathrm{ren}}(x)+\Delta\langle\hat{\varphi}_l^2\rangle(x)],$$

where the anomaly is

$$\begin{aligned}\Delta\langle\hat{\varphi}_l^2\rangle&=\lim_{x'\to x}[\mathcal{G}_{l|\mathrm{div}}-G_{\mathrm{div}|l}]\\&=\frac{1}{4\pi}\sum_{k=1}^{l}\frac{(l+k)!}{(l-k)!}\frac{1}{k}\frac{(-1)^{k+1}}{(mr)^{2k}}\\&\quad+\frac{(-1)^l}{2\pi}\sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}\frac{K_k(2mr)}{(mr)^k}.\end{aligned} \tag{3.13}$$

For example, for the first two modes the anomalies are

$$\Delta\langle\hat{\varphi}_{l=0}^2\rangle=\frac{1}{2\pi}K_0(2mr),$$

$$\Delta\langle\hat{\varphi}_{l=1}^2\rangle=\frac{1}{2\pi}\left[\frac{1}{(mr)^2}-K_0(2mr)-\frac{2}{(mr)}K_1(2mr)\right].$$

Note that the anomaly diverges logarithmically as $r\to 0$, and vanishes as $r\to\infty$.

IV. THE DIMENSIONAL-REDUCTION ANOMALY IN $\langle\hat{\Phi}^2\rangle$

In the previous section we considered the dimensional-reduction anomaly in $\langle\hat{\Phi}^2\rangle$ arising from the spherical decomposition of a scalar field in flat space. In this section we shall extend those calculations to general four-dimensional spherically symmetric spaces.

Our system consists of a massive scalar field with arbitrary coupling to the four-dimensional scalar curvature, described by Eq. (2.2) with

$$V=\xi\,{}^4R. \tag{4.1}$$

We assume that the spacetime of interest is given by the line element (2.1), which in standard spherical coordinates $y^i=(\theta,\eta)$ becomes

$$ds^2=h_{ab}(x^c)dx^a dx^b+r^2(d\theta^2+\sin^2\theta d\eta^2), \tag{4.2}$$

with $r=\rho e^{-\phi(x^c)}$.

As we saw in Sec. II, under the dimensional reduction (2.4), the quantum field $\hat{\Phi}$ reduces to a collection of effective fields $\hat{\varphi}_l$ on the two-dimensional space (2.5) with metric h_{ab}, satisfying the field equation (2.6) with induced potential

$$V_l=\xi\,{}^4R+\frac{l(l+1)}{r^2}-\Delta\phi+(\nabla\phi)^2. \tag{4.3}$$

We wish to compute the anomaly associated with renormalizing this dimensionally reduced theory versus Eq. (2.2).

A standard approach to renormalization in curved space is via the heat kernel. For the system (2.2), the heat kernel $K(X,X'|s)$ is a solution of the equation

$$FK(X,X'|s)=\frac{d}{ds}K(X,X'|s) \tag{4.4}$$

with boundary condition $K(X,X'|s=0)=\delta(X,X')$. Once the heat kernel is known for a given state, both the Green function and the effective action may be obtained using

[2]See Eq. (2.7). It is easy to verify that $\Delta\phi-(\nabla\phi)^2=0$ for the line element (3.2).

[3]Examples of the correct procedure of renormalizing using the mode-decomposed subtraction terms from four dimensions in spherically symmetric spaces can be found in [14].

$$G(X,X') = \int_0^\infty ds K(X,X'|s), \tag{4.5}$$

$$W = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\int d^4X\sqrt{g}K(X,X|s). \tag{4.6}$$

Analogous formulas hold for the dimensionally reduced theory with operator $\mathcal{F}_l$, heat kernel $\mathcal{K}_l(x,x'|s)$, Green function $\mathcal{G}_l(x,x')$ and effective action $\mathcal{W}_l$.

The advantage of the heat kernel formulation is that the divergences in both the Green function and the effective action come from the $s\to 0$ limit of the s integral, and the small-s behavior of the heat kernel is known for arbitrary curved spaces of any dimension. In particular, in four dimensions,

$$K(X,X'|s) = \frac{D^{\frac{1}{2}}}{(4\pi s)^2}\exp\left\{-m^2 s - \frac{\sigma}{2s}\right\}\sum_{i=0}^{\infty} a_i(X,X')s^i. \tag{4.7}$$

Here again $\sigma=\sigma(X,X')$ is one-half of the square of the geodesic distance between the points X and X', while $D=D(X,X')$ is the Van Vleck determinant,

$$D(X,X') = \frac{1}{\sqrt{g(X)}\sqrt{g(X')}}\det\left[-\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial X'^\nu}\sigma(X,X')\right]. \tag{4.8}$$

The a_n are the Schwinger-DeWitt coefficients for the operator F of Eq. (2.2). In the coincidence limit $X'\to X$ the first few of these are

$$a_0^{\Box-\xi^4R} = 1, \tag{4.9}$$

$$a_1^{\Box-\xi^4R} = \left(\frac{1}{6}-\xi\right){}^4R, \tag{4.10}$$

$$a_2^{\Box-\xi^4R} = \frac{1}{180}[{}^4R_{\alpha\beta\gamma\delta}{}^4R^{\alpha\beta\gamma\delta} - {}^4R_{\alpha\beta}{}^4R^{\alpha\beta} + \Box\,{}^4R] + \frac{1}{6}\left(\frac{1}{6}-\xi\right)\Box\,{}^4R + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2({}^4R)^2. \tag{4.11}$$

For the two-dimensional operator $\mathcal{F}_l$ of Eq. (2.6) we only need the Schwinger-DeWitt expansion of the heat kernel in the coincidence limit. This is

$$\mathcal{K}_l(x,x|s) = \frac{1}{4\pi s}\exp\{-m^2 s\}\left[1 + s\left(\frac{1}{6}R - V_l\right) + \cdots\right]. \tag{4.12}$$

Considering Eqs. (4.5), (4.6), it is clear that in four dimensions the divergences in $G(W)$ arise from the first two[4] (three) terms in the Schwinger-DeWitt expansion for K, while in two dimensions we need consider only the first term (first two terms) in $\mathcal{K}_l$. The anomaly in $\langle\hat{\Phi}^2\rangle$ or W can then be calculated by mode-decomposing the appropriate terms from K, comparing to the heat kernel $\mathcal{K}_l$ for the dimensionally reduced theory, and finally integrating the difference over s according to Eq. (4.5) or Eq. (4.6).

Let us begin with the anomaly in $\langle\hat{\Phi}^2\rangle$. The divergent part of the Green function in four dimensions is given by

$$G_{\rm div}(X,X') = \int_0^\infty ds K_{\rm div}(X,X'|s), \tag{4.13}$$

where $K_{\rm div}$ consists of the first two terms of Eq. (4.7):

$$K_{\rm div}(X,X'|s) = \frac{1}{(4\pi s)^2}\exp\left\{-m^2 s - \frac{\sigma}{2s}\right\} \times[\mathfrak{R}_0^{\Box-\xi^4R}(X,X') + s\mathfrak{R}_1^{\Box-\xi^4R}(X,X')]. \tag{4.14}$$

Here we use the convenient notation

$$\mathfrak{R}_n^{\Box-\xi^4R}(X,X') \equiv D^{\frac{1}{2}}(X,X')a_n^{\Box-\xi^4R}(X,X'). \tag{4.15}$$

In principle, the anomaly in $\langle\hat{\Phi}^2\rangle$ is straightforward to calculate. We mode-decompose $K_{\rm div}$ in terms of Legendre polynomials in the usual manner:

$$K_{\rm div}(X,X'|s) = \sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi rr'}P_l(\cos\lambda)K_{{\rm div}|l}(x,x'|s); \tag{4.16}$$

$$K_{{\rm div}|l}(x,x'|s) = 2\pi rr'\int_{-1}^{1} d(\cos\lambda)P_l(\cos\lambda)K_{\rm div}(X,X'|s). \tag{4.17}$$

The anomaly in $\langle\hat{\Phi}^2\rangle$ is then just the coincidence limit of the difference between the subtraction terms in two dimensions and those mode-decomposed from four dimensions, integrated over s:

$$\Delta\langle\hat{\varphi}_l^2\rangle(x) = \int_0^\infty ds[\mathcal{K}_{l|{\rm div}}(x,x|s) - K_{{\rm div}|l}(x,x|s)]. \tag{4.18}$$

We encounter a difficulty, however, when we try to perform the mode decomposition. For a general space, σ and the $a_n^{\Box-\xi^4R}$ are known only for infinitesimal separations[5] of X and X', while evaluation of the mode-decomposition integral (4.17) requires knowing σ and the $a_n^{\Box-\xi^4R}$ for finite separa-

[4] Comparing to Eq. (3.8), one sees that the integral representation used for G in the previous section was just the heat kernel representation (4.5) with Eqs. (4.7)–(4.10).

[5] In terms of momentum integrals, finite separations correspond to the low-frequency regime, where the renormalization terms are not fixed by the divergences in the theory.

tions of X, X' on the two-sphere. We proceed by determining an approximate $K_{\rm div}$ for finite separation based on the following criteria:

(1) Our approximate $K_{\rm div}$ must reduce to the known value in the flat-space limit.

(2) Our approximate $K_{\rm div}$ must respect the periodicity of the two-spheres (i.e., it must be periodic in the angular separation λ with period 2π).

In a previous case [1] in which the mode decompositions were performed over noncompact spaces, the following procedure was found to work quite well. We take $X=(x,y)$ and $X'=(x,y')$; i.e., we split the points in the y-direction only. Using the well-known short-distance expansions obtained in [15,16], σ and the $\mathfrak{R}_n^{\Box-\xi^4R}$ are expanded in powers of $(y-y')$, which is equivalent to expanding in powers of the curvature. These expansions are then substituted into $K_{\rm div}$ and, assuming small curvatures, truncated at first order in the curvature for $\Delta\langle\hat{\varphi}_l^2\rangle$ and at second order for $\Delta\mathcal{W}_l$. The mode-decomposition integrals in [1] can then be evaluated with relative ease.

In the present case, the equivalent procedure is to expand σ and the $\mathfrak{R}_n^{\Box-\xi^4R}$ in powers of λ^2, which is easily done; see Appendix B. We also take into account the periodicity of the two-spheres by converting our expansions in λ^2 into expansions in $(1-\cos\lambda)$. Defining $z=\cos\lambda$, we have

$$\lambda^2=2(1-z)+\frac{1}{3}(1-z)^2+\frac{4}{45}(1-z)^3+\cdots. \quad (4.19)$$

We then substitute Eq. (4.19) for each λ^2, truncating at the lowest order in $(1-z)$ which will yield the correct flat-space limit. This replacement of λ^2 by a finite series in $(1-z)$ means that our expansions are only modified for large angular separations, where the renormalization terms are inherently ambiguous. Our choice simply corresponds to a natural extension of the flat-space heat kernel which respects the periodicity of the two-spheres for large angular separations. For more details, see Appendix B.

Using this procedure, one finds that to first order in the curvature

$$2\sigma=2r^2(1-z)+\frac{r^2}{3}[1-r^2(\nabla\phi)^2](1-z)^2, \quad (4.20)$$

$$\mathfrak{R}_0^{\Box-\xi^4R}=1+\frac{1}{6}{}^4R_{\theta\theta}(1-z), \quad (4.21)$$

$$\mathfrak{R}_1^{\Box-\xi^4R}=\left(\frac{1}{6}-\xi\right){}^4R. \quad (4.22)$$

Inserting these expansions into Eq. (4.7) yields our approximation for the ''divergent'' part of the four-dimensional heat kernel,

$$\begin{aligned}K_{\rm div}(X,X'|s)=&\frac{1}{(4\pi s)^2}\exp\left\{-m^2s-\frac{r^2}{2s}(1-z)\right\}\\&\times\left[1+s\left(\frac{1}{6}-\xi\right){}^4R+\frac{1}{6}{}^4R_{\theta\theta}(1-z)\right.\\&\left.-\frac{r^2}{12s}[1-r^2(\nabla\phi)^2](1-z)^2\right]. \quad (4.23)\end{aligned}$$

The mode decomposition (4.17) of this $K_{\rm div}$ then boils down to evaluating the integrals

$$J_{ln}\equiv-p\int_{-1}^{1}dz\,P_l(z)e^{p(1-z)}(1-z)^n, \quad (4.24)$$

where $p\equiv-r^2/2s$ is a dimensionless parameter, and n is an integer. The integrals for $n\neq0$ can be obtained from the $n=0$ result (3.9), (3.10) used in the flat-space case by differentiating with respect to p, yielding

$$\begin{aligned}J_{ln}=&\sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}\frac{1}{(-2p)^k}\left[\frac{(-1)^k2^n}{(-2p)^n}\frac{(k+n)!}{k!}\right.\\&\left.-(-1)^le^{2p}\sum_{\alpha=0}^{n}\frac{2^n}{(-2p)^\alpha}\frac{(k+\alpha)!}{k!}\frac{n!}{\alpha!(n-\alpha)!}\right]. \quad (4.25)\end{aligned}$$

The mode-decomposed heat kernel subtraction terms for a general four-dimensional spherically symmetric spacetime are then

$$\begin{aligned}K_{{\rm div}|l}(x,x|s)=&\frac{e^{-m^2s}}{4\pi s}\left[J_{l0}+s\left(\frac{1}{6}-\xi\right){}^4RJ_{l0}+\frac{1}{6}{}^4R_{\theta\theta}J_{l1}\right.\\&\left.-\frac{r^2}{12s}[1-r^2(\nabla\phi)^2]J_{l2}\right]. \quad (4.26)\end{aligned}$$

The first term in Eq. (4.26) is the mode decomposition for flat space, while the other terms carry the contributions due to the curvature. Meanwhile, the various parts of the J_{ln} fulfill several roles. First, the $k\neq0$ terms in Eq. (4.25) are associated with the centrifugal potential $l(l+1)/r^2$ induced by the mode decomposition. This potential is ignored in the renormalization in two dimensions, since only the first (potential-independent) term in the Schwinger-DeWitt expansion of the heat kernel contributes divergences to the two-dimensional Green function. Second, then terms in Eq. (4.25) proportional to $e^{2p}=e^{-r^2/s}$ enforce a Dirichlet boundary condition at $r=0$, which is required if the four-dimensional subtraction term is to be finite there [see Eq. (4.16)].

These results are to be compared with the subtraction term in two dimensions, which consists of the first term of Eq. (4.12):

$$\mathcal{K}_{l|{\rm div}}(x,x|s)=\frac{e^{-m^2s}}{4\pi s}. \quad (4.27)$$

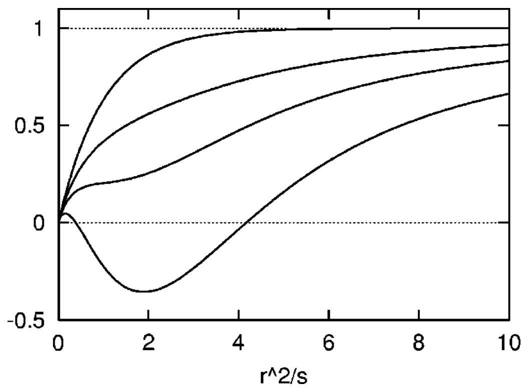

FIG. 1. $K_{\mathrm{div}|l=0}(x,x|s)$ in Schwarzschild space for fixed s and $M/\sqrt{s}=0,2,4,8$ from top to bottom. The common factor $e^{-m^2 s}/(4\pi s)$ has been removed. The two-dimensional subtraction term $\mathcal{K}_{l|\mathrm{div}}$ would be a horizontal line at 1 on this plot.

In contrast to $K_{\mathrm{div}|l}$, $\mathcal{K}_{l|\mathrm{div}}$ is independent of both the position, the two-metric h_{ab}, and the mode number l. It matches just the first term in the $k=0$ contribution to the flat space part of $K_{\mathrm{div}|l}$.

As an example, let us consider a quantum field in Schwarzschild space, for which ${}^4R=0$, ${}^4R_{\mu\nu}=0$, and

$$[1-r^2(\nabla\phi)^2]=\frac{2M}{r}, \tag{4.28}$$

where M is the black-hole mass. Figure 1 shows plots of $K_{\mathrm{div}|l=0}(x,x|s)$ for fixed s and various values of $M/\sqrt{s}$. Note that large values of M cause the mode-decomposed subtraction terms to become negative.

The anomaly in $\langle\hat{\Phi}^2\rangle$ can now be found by integrating the difference of $\mathcal{K}_{l|\mathrm{div}}$ and $K_{\mathrm{div}|l}$ as in Eq. (4.18). We find

$$\Delta\langle\hat{\varphi}_l^2\rangle=\frac{1}{2\pi}\Bigg[-I[m^2 sJ_{l0}]-\frac{1}{m^2}\left(\frac{1}{6}-\xi\right){}^4RI[(m^2s)^2J_{l0}] -\frac{1}{6}{}^4R_{\theta\theta}I[m^2sJ_{l1}]+\frac{(mr)^2}{12}[1-r^2(\nabla\phi)^2]I[J_{l2}]\Bigg], \tag{4.29}$$

where

$$I[(m^2s)^tJ_{ln}]\equiv\sum_{k=2-t-n}^{l}\frac{(l+k)!}{k!(l-k)!}2^n\times\left[\frac{1}{2}\frac{(-1)^k}{(mr)^{2n+2k}}\frac{(k+n)!}{k!}(k+t+n-2)!\right] -(-1)^l\sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}2^n\times\left[\sum_{\alpha=0}^{n}\frac{(k+\alpha)!}{k!}\frac{n!}{\alpha!(n-\alpha)!}\times\frac{K_{k+t+\alpha-1}(2mr)}{(mr)^{k-t+\alpha+1}}\right]. \tag{4.30}$$

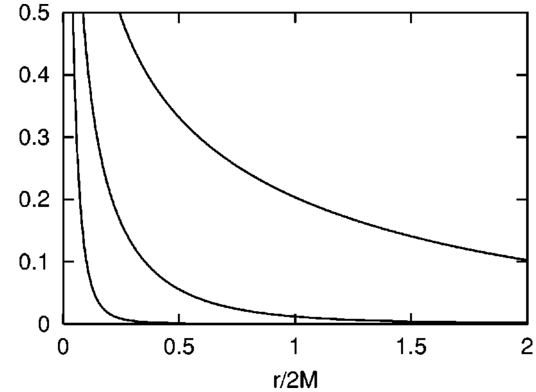

FIG. 2. $\Delta\langle\hat{\varphi}_{l=0}^2\rangle$ in Schwarzschild space for $mM=0.1,1,10$, from top to bottom.

(The $I[(m^2s)^tJ_{ln}]$ result from integrating terms of the form $(m^2s)^{t-2}J_{ln}$ over s.) For example, in Schwarzschild space the anomaly for the $l=0$ mode is

$$\Delta\langle\hat{\varphi}_{l=0}^2\rangle=\frac{1}{2\pi}K_0(2mr)+\frac{M}{3\pi r}\Bigg[\frac{1}{(mr)^2}-\frac{2}{mr}K_1(2mr) -2K_0(2mr)-mrK_1(2mr)\Bigg]. \tag{4.31}$$

Plots of $\Delta\langle\hat{\varphi}_{l=0}^2\rangle$ for various values of mM are shown in Fig. 2. Note that the anomaly in $\langle\hat{\Phi}^2\rangle$ generally diverges at any point x^a such that $r(x^a)=0$, while for asymptotically flat spaces it vanishes as $r\to\infty$.

V. THE DIMENSIONAL-REDUCTION ANOMALY IN THE EFFECTIVE ACTION

In the previous section we calculated the dimensional-reduction anomaly in $\langle\hat{\Phi}^2\rangle$ for a general four-dimensional spherically symmetric space. We now use the same procedure to determine the anomaly in the effective action, denoted by $\Delta\mathcal{W}_l$ in Eq. (2.16). Functional differentiation of $\Delta\mathcal{W}_l$ with respect to the metric h_{ab} would then give the corresponding anomaly $\Delta\langle\hat{\mathcal{T}}_{\mu\nu}\rangle_l$ in the stress tensor in Eq. (2.17).

For the four-dimensional effective action (4.6) the divergent part of the heat kernel consists of the first three terms of Eq. (4.7):

$$K_{\mathrm{div}}(X,X'|s)=\frac{1}{(4\pi s)^2}\exp\left\{-m^2s-\frac{\sigma}{2s}\right\}\times[\mathfrak{R}_0^{\Box-\xi^4R}+s\mathfrak{R}_1^{\Box-\xi^4R}+s^2\mathfrak{R}_2^{\Box-\xi^4R}]. \tag{5.1}$$

As in the previous section, we split the points X, X' in the angular direction only. Then one can write

$$2\sigma = 2r^2[(1-z)+u(1-z)^2+v(1-z)^3+\cdots], \quad (5.2)$$

$$\mathfrak{R}_n^{\Box-\xi^4R} = \mathfrak{R}_{n(0)}^{\Box-\xi^4R} + \mathfrak{R}_{n(1)}^{\Box-\xi^4R}(1-z) + \mathfrak{R}_{n(2)}^{\Box-\xi^4R}(1-z)^2 + \cdots, \quad (5.3)$$

where $z=\cos\lambda$. From the calculations for the anomaly in $\langle\hat{\Phi}^2\rangle$ we have seen that $u=\frac{1}{6}[1-r^2(\nabla\phi)^2]$, $\mathfrak{R}_{0(0)}^{\Box-\xi^4R}=1$, $\mathfrak{R}_{0(1)}^{\Box-\xi^4R}=\frac{1}{6}\,{}^4R_{\theta\theta}$, and $\mathfrak{R}_{1(0)}^{\Box-\xi^4R}=(\frac{1}{6}-\xi)\,{}^4R$. The other $\mathfrak{R}_{n(k)}^{\Box-\xi^4R}$ and v are found in Appendix B. Inserting these expansions into Eq. (5.1) and truncating at second order in the curvature, we find

$$K_{\rm div}(X,X'|s) = \frac{1}{(4\pi s)^2}\exp\left\{-m^2s-\frac{r^2}{2s}(1-z)\right\}\left[\{1+s\,\mathfrak{R}_{1(0)}^{\Box-\xi^4R}+s^2\mathfrak{R}_{2(0)}^{\Box-\xi^4R}\}+\{\mathfrak{R}_{0(1)}^{\Box-\xi^4R}+s\,\mathfrak{R}_{1(1)}^{\Box-\xi^4R}\}(1-z)\right.$$
$$\left.+\left\{\mathfrak{R}_{0(2)}^{\Box-\xi^4R}-\frac{r^2u}{2s}-\frac{r^2u}{2}\mathfrak{R}_{1(0)}^{\Box-\xi^4R}\right\}(1-z)^2+\left\{-\frac{r^2u}{2s}\mathfrak{R}_{0(1)}^{\Box-\xi^4R}-\frac{r^2v}{2s}\right\}(1-z)^3+\frac{r^4u^2}{8s^2}(1-z)^4\right]. \quad (5.4)$$

The decomposition of the heat kernel subtraction terms (5.4) is done in the same manner as in the previous section. Employing the definition (4.17) of the spherical decomposition and using the functions J_{ln} of Eqs. (4.24), (4.25), we obtain

$$K_{{\rm div}|l}(x,x|s) = \lim_{x'\to x} 2\pi r^2\int_{-1}^{1} d(\cos\lambda)P_l(\cos\lambda)K_{\rm div}(X,X'|s)$$
$$= \frac{e^{-m^2s}}{4\pi s}\left[\{1+s\mathfrak{R}_{1(0)}^{\Box-\xi^4R}+s^2\mathfrak{R}_{2(0)}^{\Box-\xi^4R}\}J_{l0}+\{\mathfrak{R}_{0(1)}^{\Box-\xi^4R}+s\mathfrak{R}_{1(1)}^{\Box-\xi^4R}\}J_{l1}+\left\{\mathfrak{R}_{0(2)}^{\Box-\xi^4R}-\frac{r^2u}{2s}-\frac{r^2u}{2}\mathfrak{R}_{1(0)}^{\Box-\xi^4R}\right\}J_{l2}\right.$$
$$\left.+\left\{-\frac{r^2v}{2s}-\frac{r^2u}{2s}\mathfrak{R}_{0(1)}^{\Box-\xi^4R}\right\}J_{l3}+\frac{r^4u^2}{8s^2}J_{l4}\right]. \quad (5.5)$$

Meanwhile, the divergences in the effective action for the two-dimensional theory (2.6) arise from the first two terms of Eq. (4.12):

$$\mathcal{K}_{l|{\rm div}}(x,x|s) = \frac{e^{-m^2s}}{4\pi s}\left[1+s\left(\frac{1}{6}R-V_l\right)\right]. \quad (5.6)$$

The anomaly in the effective action is then found by integrating the difference of Eqs. (5.5), (5.6) as in Eq. (4.6):

$$\Delta\mathcal{W}_l = -\frac{1}{2}\int d^2x\sqrt{h}\int_0^\infty\frac{ds}{s}[\mathcal{K}_{l|{\rm div}}(x,x|s)-K_{{\rm div}|l}(x,x|s)]$$
$$= \frac{m^2}{4\pi}\int d^2x\sqrt{h}\left\{I[J_{l0}]+\frac{1}{m^2}\mathfrak{R}_{1(0)}^{\Box-\xi^4R}I[m^2sJ_{l0}]+\frac{1}{m^4}\mathfrak{R}_{2(0)}^{\Box-\xi^4R}I[m^4s^2J_{l0}]+\mathfrak{R}_{0(1)}^{\Box-\xi^4R}I[J_{l1}]+\frac{1}{m^2}\mathfrak{R}_{1(1)}^{\Box-\xi^4R}I[m^2sJ_{l1}]\right.$$
$$\left.+\left(\mathfrak{R}_{0(2)}^{\Box-\xi^4R}-\frac{r^2u}{2}\mathfrak{R}_{1(0)}^{\Box-\xi^4R}\right)I[J_{l2}]-\frac{(mr)^2u}{2}I\left[\frac{1}{m^2s}J_{l2}\right]-\frac{(mr)^2}{2}(u\mathfrak{R}_{0(1)}^{\Box-\xi^4R}+v)I\left[\frac{1}{m^2s}J_{l3}\right]+\frac{(mr)^4u^2}{8}I\left[\frac{1}{m^4s^2}J_{l4}\right]\right\}. \quad (5.7)$$

The $I[(m^2s)^tJ_{ln}]$ are given by Eq. (4.30). Using Eq. (5.7) and the values of u, v, and the $\mathfrak{R}_{n(k)}^{\Box-\xi^4R}$ given in Appendix B, one can compute the anomalous contribution to the stress tensor.

VI. CONCLUSIONS

In a D-dimensional spacetime which can be foliated by n-dimensional homogeneous subspaces, a field can be decomposed in terms of modes on the subspaces. This effectively converts the system from a single quantum field in D dimensions to a collection of fields in $(D-n)$ dimensions. Quantities of interest for the original theory, such as the expectation value of the square of the field operator and the effective action, can then be written as sums of the corresponding objects from the dimensionally reduced theories. This relationship breaks down under renormalization, however, so that renormalized expectation values can be obtained by summing their lower-dimensional counterparts only if the contribution for each mode is modified by adding an anomalous contribution. This effect is the dimensional reduction anomaly.

We have explicitly calculated the anomalous contributions to the expectation value of the square of the field operator and the effective action for the case of a massive scalar field propagating in a general four-dimensional spherically symmetric space. We have seen that the anomaly arises from several sources. One is the Dirichlet boundary condition imposed at $r=0$ due to the change in topology inherent in the spherical reduction of the spacetime. Other contributions are more local in nature, arising from the dimension-dependent contributions of the curvature and field potential to divergences. The resulting anomaly terms are constructed from the curvature, the dilaton field, and their covariant derivatives, and cannot be eliminated by further finite renormalization.

The anomalies calculated in this paper may be of importance to recent attempts to calculate the stress tensor and Hawking radiation in black-hole spacetimes using quantum fields in two dimensions [2,7]. These attempts are based on the dimensional reduction of a massless minimally-coupled quantum field in a Schwarzschild spacetime, followed by renormalization in two dimensions. We have seen, however, that the contributions of the dimensionally reduced fields should be modified by adding the corresponding anomaly term. We intend to return to a discussion of this interesting topic in a future publication.

ACKNOWLEDGMENTS

The author would like to thank Valeri Frolov and Andrei Zelnikov for useful discussions. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX A: SPHERICAL DECOMPOSITION OF CURVATURES

Consider a line element of the form

$$ds^2=g_{\mu\nu}dX^\mu dX^\nu=h_{ab}dx^a dx^b+\rho^2 e^{-2\phi}\omega_{ij}dy^i dy^j, \tag{A1}$$

where $h_{ab}=h_{ab}(x^c)$ is an arbitrary two-dimensional metric and $\omega_{ij}=\omega_{ij}(y^k)$ is the metric of a two-sphere. The dilaton ϕ is a function of the x^a only, and ρ is a constant with the dimensions of length. The radius of a two-sphere of fixed x^a is $r=\rho e^{-\phi}$.

We wish to decompose our field theory in terms of modes on the two-sphere. This requires rewriting four-dimensional geometric quantities like the curvatures in terms of the corresponding curvatures for the metric h.

Our notational conventions are as follows: four-dimensional covariant derivatives are denoted by $()_{;a}$, while $\Box$ is understood to represent the d'Alembertian with respect to g. Meanwhile, ∇, $()_{|a}$ and Δ are the two-dimensional covariant derivatives and d'Alembertian calculated using the metric h_{ab}. For the dilaton ϕ we shall understand ϕ_a, ϕ_{ab}, etc. to denote multiple two-dimensional covariant derivatives of ϕ. For example, the four-dimensional d'Alembertian of an angle-independent scalar S decomposes to

$$\Box S=\Delta S-2\nabla\phi\cdot\nabla S. \tag{A2}$$

In particular,

$$\Box\phi=\Delta\phi-2(\nabla\phi)^2. \tag{A3}$$

For the given line element, the nonvanishing Christoffel symbols are

$${}^4\Gamma^a_{bc}[g]={}^2\Gamma^a_{bc}[h], \tag{A4}$$

$${}^4\Gamma^a_{ij}[g]=\phi^a g_{ij}, \tag{A5}$$

$${}^4\Gamma^i_{ja}[g]=-\phi_a\delta^i_j, \tag{A6}$$

$${}^4\Gamma^k_{ij}[g]={}^2\Gamma^k_{ij}[\omega]. \tag{A7}$$

Selecting coordinates (θ,η) on the two-spheres, where

$$d\omega_{ij}y^i y^j=d\theta^2+\sin^2\theta d\eta^2, \tag{A8}$$

one finds

$${}^4\Gamma^\theta_{\eta\eta}[g]=-\sin\theta\cos\theta, \quad {}^4\Gamma^\eta_{\eta\theta}[g]=\frac{\cos\theta}{\sin\theta}. \tag{A9}$$

For convenience, we define the following commonly occurring functions of the dilaton field:

$$A=1-r^2(\nabla\phi)^2, \tag{A10}$$

$$B=\Delta\phi-2(\nabla\phi)^2, \tag{A11}$$

$$T_{ab}=\phi_{ab}-\phi_a\phi_b, \tag{A12}$$

$$T=h^{ab}T_{ab}=\Delta\phi-(\nabla\phi)^2. \tag{A13}$$

Since the two-sphere metric has constant curvature ${}^2R[\omega]=2$, explicit reference to it may be dropped. Henceforth we assume all curvatures to be with respect to the two-dimensional metric h_{ab} unless explicitly labeled otherwise. Using this notation, one can show that the only nonvanishing components of the four-dimensional curvatures are

$${}^4R_{abcd}[g]=\frac{1}{2}R(h_{ac}h_{bd}-h_{ad}h_{bc}), \tag{A14}$$

$${}^4R_{aibj}[g]=g_{ij}T_{ab}, \tag{A15}$$

$${}^4R_{ijkm}[g]=\frac{A}{r^2}(g_{ik}g_{jm}-g_{im}g_{jk}), \tag{A16}$$

$${}^4R_{ab}[g]=\frac{1}{2}Rh_{ab}+2T_{ab}, \tag{A17}$$

$$^4R_{ij}[g]=g_{ij}\left[\frac{1}{r^2}+B\right], \tag{A18}$$

$$^4R[g]=R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}, \tag{A19}$$

while the only nonvanishing $^4R_{\alpha\beta;\gamma}$ are

$$^4R_{ab;c}[g]=\frac{1}{2}h_{ab}R_{|c}+2T_{ab|c}, \tag{A20}$$

$$^4R_{am;n}[g]=g_{mn}\left[\left(-\frac{1}{2}R+\frac{1}{r^2}+B\right)\phi_a-2T_{ab}\phi^b\right], \tag{A21}$$

$$^4R_{mn;a}[g]=g_{mn}\left(\frac{1}{r^2}+B\right)_{|a}. \tag{A22}$$

Also,

$$^4R_{mn;ab}[g]=g_{mn}\left(\frac{1}{r^2}+B\right)_{|ab}, \tag{A23}$$

$$^4R_{mn;jk}[g]=-(g_{km}g_{nj}+g_{kn}g_{mj})\left[\left(-\frac{1}{2}R+\frac{1}{r^2}+B\right)(\nabla\phi)^2-2T_{ab}\phi^a\phi^b\right]-g_{jk}g_{mn}\left(\frac{1}{r^2}+B\right)_{|a}\phi^a, \tag{A24}$$

$$\Box\,^4R_{mn}[g]=g_{mn}\left\{[\Delta-2\nabla\phi\cdot\nabla]\left(\frac{1}{r^2}+B\right)+R(\nabla\phi)^2-2\left(\frac{1}{r^2}+B\right)(\nabla\phi)^2+4T_{ab}\phi^a\phi^b\right\}, \tag{A25}$$

$$^4R_{;a}=\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)_{|a}, \tag{A26}$$

$$^4R_{;ab}=\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)_{|ab}, \tag{A27}$$

$$^4R_{;mn}=-g_{mn}\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)_{|a}\phi^a, \tag{A28}$$

$$\Box^4R=[\Delta-2\nabla\phi\cdot\nabla]\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right). \tag{A29}$$

APPENDIX B: POINT SPLITTING

It will be necessary to write short-distance expansions for σ and the $D^{1/2}a_n$ for X and X' separated along the two-spheres. We follow a method similar to that developed in [1]. Without loss of generality we take the points to be split in the θ direction only, with angular separation $\lambda=\theta-\theta'$. Our procedure will be to calculate the desired quantities first as expansions in powers of λ^2, and then to convert them to expansions in powers of $(1-\cos\lambda)$ for use in the mode-decomposition calculations.

We take as our ansatz for the geodetic interval σ

$$2\sigma(x,y;x',y')=(\tilde{r}\lambda)^2+U(\tilde{x})(\tilde{r}\lambda)^4+V(\tilde{x})(\tilde{r}\lambda)^6+\cdots, \tag{B1}$$

where $\tilde{x}\equiv\frac{1}{2}(x+x')$. Taking the derivative of $\tilde{\sigma}$ with respect to each of the coordinates and requiring $\sigma=\frac{1}{2}g^{\alpha\beta}\sigma_\alpha\sigma_\beta$ in the coincidence limit, one can show that

$$U(x)=-\frac{1}{12}(\nabla\phi)^2, \tag{B2}$$

$$V(x)=\frac{1}{90}(\nabla\phi)^4-\frac{1}{120}\phi^a\phi^b\phi_{ab}, \tag{B3}$$

and

$$(\sigma^\theta)^2=\lambda^2\left[1-\frac{1}{3}r^2(\nabla\phi)^2\lambda^2+r^4\left(\frac{17}{180}(\nabla\phi)^4-\frac{1}{20}\phi^a\phi^b\phi_{ab}\right)\lambda^4+\cdots\right], \tag{B4}$$

$$\sigma^\eta=0, \tag{B5}$$

$$\sigma^a=-\frac{1}{2}\phi^a(r\lambda)^2+\left[-\frac{1}{24}\phi^{ab}\phi_b+\frac{1}{12}(\nabla\phi)^2\phi^a\right](r\lambda)^4+\cdots. \tag{B6}$$

The expansion (B1) for σ can be converted into one in terms of $(1-\cos\lambda)$ using

$$\lambda^2=2(1-z)+\frac{1}{3}(1-z)^2+\frac{4}{45}(1-z)^3+\cdots, \tag{B7}$$

where $z\equiv\cos\lambda$. Defining the functions $u(x)$, $v(x)$ by

$$2\sigma(x,y;x,y')=2r^2[(1-z)+u(x)(1-z)^2+v(x)(1-z)^3+\cdots], \tag{B8}$$

we obtain

$$u(x)=\frac{1}{6}[1-r^2(\nabla\phi)^2], \tag{B9}$$

$$v(x)=\frac{2}{45}\left[1-\frac{5}{4}r^2(\nabla\phi)^2+r^4(\nabla\phi)^4-\frac{3}{8}r^4\nabla\phi\cdot\nabla[(\nabla\phi)^2]\right].\tag{B10}$$

Combining Eqs. (B4)-(B7) with the results of Appendix A and the short-distance expansions of [15,16], one can derive expansions for the $D^{1/2}a_n$ in powers of $(1-z)$. Writing

$$D^{1/2}a_n^{\Box-\xi^4R}=\mathfrak{R}_n^{\Box-\xi^4R}=\mathfrak{R}_{n(0)}^{\Box-\xi^4R}+\mathfrak{R}_{n(1)}^{\Box-\xi^4R}(1-z)+\mathfrak{R}_{n(2)}^{\Box-\xi^4R}(1-z)^2+\cdots,\tag{B11}$$

one can show that

$$\mathfrak{R}_{0(0)}^{\Box-\xi^4R}=1,\tag{B12}$$

$$\mathfrak{R}_{0(1)}^{\Box-\xi^4R}=\frac{1}{6}(1+r^2B),\tag{B13}$$

$$\mathfrak{R}_{0(2)}^{\Box-\xi^4R}=\frac{1}{90}A^2+\frac{1}{72}(1+r^2B)^2+\frac{1}{36}(1+r^2B)[1-4r^2(\nabla\phi)^2]+\frac{r^4}{180}\left[\frac{3}{2}R(\nabla\phi)^2+6T_{ab}\phi^a\phi^b+2T_{ab}T^{ab}+12\left(\frac{1}{r^2}+B\right)(\nabla\phi)^2+6\left(\frac{1}{r^2}+B\right)_{|a}\phi^a\right],\tag{B14}$$

$$\mathfrak{R}_{1(0)}^{\Box-\xi^4R}=\left(\frac{1}{6}-\xi\right)\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right),\tag{B15}$$

$$\begin{aligned}\mathfrak{R}_{1(1)}^{\Box-\xi^4R}=&\frac{1}{6}\left(\frac{1}{6}-\xi\right)[(1+r^2B)+r^2\nabla\phi\cdot\nabla]\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)+\frac{r^2}{180}\Bigg[RT+3R(\nabla\phi)^2+8T_{ab}T^{ab}+12T_{ab}\phi^a\phi^b\\&+\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)_{|a}\phi^a+3[\Delta-2\nabla\phi\cdot\nabla]\left(\frac{1}{r^2}+B\right)-6\left(\frac{1}{r^2}+B\right)(\nabla\phi)^2\Bigg]\\&+\frac{1}{90r^2}[2A^2+A(1+r^2B)-2(1+r^2B)^2],\end{aligned}\tag{B16}$$

$$\begin{aligned}\mathfrak{R}_{2(0)}^{\Box-\xi^4R}=&\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)^2+\frac{1}{6}\left(\frac{1}{6}-\xi\right)[\Delta-2\nabla\phi\cdot\nabla]\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)\\&+\frac{1}{180}\left[[\Delta-2\nabla\phi\cdot\nabla]\left(R+4\Delta\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)+\frac{1}{2}R^2-2RT+4T_{ab}T^{ab}+\frac{4}{r^4}A^2-\frac{2}{r^4}(1+r^2B)^2\right].\end{aligned}\tag{B17}$$

It is easily verified that for flat spacetime each of the $\mathfrak{R}_{n(k)}^{\Box-\xi^4R}$ vanishes, except for $\mathfrak{R}_{0(0)}^{\Box-\xi^4R}$.

[1] V. Frolov, P. Sutton, and A. Zelnikov, Phys. Rev. D **61**, 024021 (2000).
[2] V. Mukhanov, A. Wipf, and A. Zelnikov, Phys. Lett. B **332**, 283 (1994).
[3] T. Chiba and M. Siino, Mod. Phys. Lett. A **12**, 709 (1997).
[4] R. Bousso and S. Hawking, Phys. Rev. D **56**, 7788 (1997).
[5] W. Kummer, H. Liebl, and D.V. Vassilevich, Mod. Phys. Lett. A **12**, 2683 (1997).
[6] R. Balbinot and A. Fabbri, Phys. Rev. D **59**, 044031 (1999); Phys. Lett. B **459**, 112 (1999); W. Kummer, H. Liebl, and D.V. Vassilevich, Phys. Rev. D **58**, 108501 (1998); W. Kummer and D.V. Vassilevich, Ann. Phys. (Leipzig) **8**, 801 (1999); Phys. Rev. D **60**, 084021 (1999); S. Nojiri and S.D. Odintsov, Mod. Phys. Lett. A **12**, 2083 (1997); Phys. Rev. D **57**, 2363 (1998); A. Mikovic and V. Radovanovic, Class. Quantum Grav. **15**, 827 (1998); F.C. Lombardo, F.D. Mazzitelli, and J.G. Russo, Phys. Rev. D **59**, 064007 (1999).
[7] S. Nojiri and S.D. Odintsov, Phys. Lett. B **463**, 57 (1999).
[8] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[9] E. Elizalde, L. Vanzo, and S. Zerbini, Commun. Math. Phys. **194**, 613 (1998).

[10] J.S. Dowker, hep-th/9803200.
[11] T.S. Evans, Phys. Lett. B **457**, 127 (1999).
[12] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series*, Vol. 1. *Elementary Functions*, and Vol. 2. *Special Functions* (Gordon and Breach, London, 1986).
[13] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1994).
[14] P. Candelas, Phys. Rev. D **21**, 2185 (1980); P.R. Anderson, *ibid.* **41**, 1152 (1990).
[15] S.M. Christensen, Phys. Rev. D **14**, 2490 (1976).
[16] S.M. Christensen, Phys. Rev. D **17**, 946 (1978).